Note to the reader: *This is a reproduction of notes I took in METR 4433, Mesoscale Meteorology, at the University of Oklahoma. I have expanded the derivation so that it might make a little more sense. If you find an error, please tell me! ck*

The Pressure Perturbation Equation: Exposed!

The rotational dynamics of supercell storms have a lot to do with the pressure perturbations created by the air flow. It is this effect that makes supercells special.

Phase 1: Derivation

Take the divergence of the vector equation of motion:

$$\nabla \cdot \frac{d\vec{V}}{dt} = \nabla \cdot \left(-\frac{1}{\rho_o} \nabla p' \right) + \nabla \cdot B\hat{k},$$

or, in component form:

First term: $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) = -\frac{1}{\rho_o}\frac{\partial^2 p'}{\partial x^2}$$

Second term: $$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) = -\frac{1}{\rho_o}\frac{\partial^2 p'}{\partial y^2}$$

Third term: $$\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) = -\frac{1}{\rho_o}\frac{\partial^2 p'}{\partial z^2} + \frac{\partial B}{\partial z}.$$

If we assume that the vertical wind field is *linear*, then second derivative terms vanish. If we assume also that the wind field is unchanging with time (I know, a stretch, given a supercell in the neighborhood), then terms involving ∂t can also be neglected.

Take the remaining derivatives and group terms, as follows:

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial z} = -\frac{1}{\rho_o}\nabla^2 p' + \frac{\partial B}{\partial z}$$

Notice that we have squared terms, and some terms that are the same (i.e., multiples of a term).

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + 2\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y} \right) = -\frac{1}{\rho_o}\nabla^2 p' + \frac{\partial B}{\partial z}$$

Now rewrite the equation in terms of the pressure perturbation, which is what we want!

$$\nabla^2 p' = -\rho_o\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \right] - 2\rho_o\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y} \right) + \rho_o\frac{\partial B}{\partial z}$$

A B C

On the right-hand side, group (A) represents the "fluid extension" terms; (b) are the non-linear terms (obviously), and (c) is the vertical buoyancy gradient.

Phase 2: Mean and Perturbation Flows

Now, let us partition the winds between an environmental "mean" wind component and the thunderstorm-induced perturbation winds:

$$u(x,y,z,t)=\overline{u}(z)+u'(x,y,z,t)$$
$$v(x,y,z,t)=\overline{v}(z)+v'(x,y,z,t)$$
$$w(x,y,z,t)=0+w'(x,y,z,t)$$

We assume the mean flow is horizontal. That is, all vertical motion is attributed to the thunderstorm; more specifically, the vertical motion induced by the thunderstorm is a result of the pressure perturbations within the storm.

Now, substitute the mean and perturbation relationships into the pressure perturbation equation we have so far developed. Note that in terms A, only the perturbations will be left (in u and v, the mean flow does not vary in the horizontal; in w, there is no mean flow!). In terms B, there will be terms with mean flow (can you see which ones?). After this process, the equation becomes:

$$\begin{aligned}\nabla^2 p' = &-\rho_o\left[\left(\frac{\partial u'}{\partial x}\right)^2+\left(\frac{\partial v'}{\partial y}\right)^2+\left(\frac{\partial w'}{\partial z}\right)^2\right]\\ &-2\rho_o\left(\frac{\partial v'}{\partial x}\frac{\partial u'}{\partial y}+\frac{\partial \overline{u}}{\partial z}\frac{\partial w'}{\partial x}+\frac{\partial u'}{\partial z}\frac{\partial w}{\partial x}+\frac{\partial \overline{v}}{\partial z}\frac{\partial w'}{\partial y}+\frac{\partial v'}{\partial z}\frac{\partial w'}{\partial y}\right)\\ &+\rho_o\frac{\partial B}{\partial z}\end{aligned}$$

Rewrite the terms involving "mean wind shear" by themselves, and now group all the non-linear terms together. This gives

$$\begin{aligned}\nabla^2 p' = &-2\rho_o\left(\frac{\partial \overline{u}}{\partial z}\frac{\partial w'}{\partial x}+\frac{\partial \overline{v}}{\partial z}\frac{\partial w'}{\partial y}\right)\\ &-\rho_o\left[\left(\frac{\partial u'}{\partial x}\right)^2+\left(\frac{\partial v'}{\partial y}\right)^2+\left(\frac{\partial w'}{\partial z}\right)^2+2\frac{\partial v'}{\partial x}\frac{\partial u'}{\partial y}+2\frac{\partial u'}{\partial z}\frac{\partial w}{\partial x}+2\frac{\partial v'}{\partial z}\frac{\partial w'}{\partial y}\right]\\ &+\rho_o\frac{\partial B}{\partial z}\end{aligned}$$

Phase 3. Application

Updraft enhancement in rotation thunderstorms and cell division

This is an elliptic diagnostic equation for pressure perturbation, *p'*. We can divide the total perturbation pressure as follows:

$$p' = p'_{dyn} + p'_{buoyancy}, \text{ or}$$
$$p' = p'_{linear} + p'_{non-linear} + p'_{buoyancy}$$

Each part is attributed to certain terms on the right hand side of the pressure perturbation equation.

We saw earlier that a strong updraft in an environment of significant vertical shear produces a pair of counter-rotating vortices. Consider the $p'_{non\text{-}linear}$ term only. It can be verified that the three shear terms can be written in the following form:

$$2\frac{\partial v'}{\partial x}\frac{\partial u'}{\partial y} + 2\frac{\partial u'}{\partial z}\frac{\partial w'}{\partial x} + 2\frac{\partial v'}{\partial z}\frac{\partial w'}{\partial y}$$
$$= \frac{1}{2}\left[\left(\frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y}\right)^2 + \left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right)^2 + \left(\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right)^2 - \left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right)^2 - \left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right)^2 - \left(\frac{\partial v'}{\partial z} - \frac{\partial w'}{\partial y}\right)^2\right]$$

If we assume pure rotation (so there is no divergence nor deformation) and ignore the extension terms (the 'squared' terms in $p'_{non\text{-}linear}$; i.e., we are looking at the effect of rotation only), then

$$\nabla^2 p'_{non-linear} = \frac{\rho_o}{2}\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right)^2 = \frac{\rho_o}{2}\zeta'^2$$

Since the left-hand side contains a Laplacian operator, $\nabla^2 p'_{NL} \propto -p'_{NL}$, therefore

$$\boxed{p'_{NL} \propto -\zeta'^2}$$

Both cyclonic and anticyclonic rotation produce negative pressure perturbation. The low pressure center is actually *required* to have a PGF that balances the centrifugal force!

Negative p'_{NL} will be largest where rotation is the strongest, which is usually at the mid-levels of thunderstorms. The magnitude of the pressure perturbation there can be as high as 2-4 mb.

Earlier figures show that because of tilting, vertical rotation is the strongest at the flanks of the updraft, and the negative *p'* at the mid-levels creates an updraft pressure gradient force that promotes new updrafts there—**a dynamic cause for cell-splitting.**

Now, let's consider the fluid extension terms.

$$\nabla^2 p' = -\rho_o\left[\left(\frac{\partial u'}{\partial x}\right)^2 + \left(\frac{\partial v'}{\partial y}\right)^2 + \left(\frac{\partial w'}{\partial z}\right)^2 + \ldots\right]$$

The right-hand side will always be negative; therefore, the fluid extension terms always create a positive pressure perturbation. Where is stretching the largest? At the low and high levels of the atmosphere.

So we have

$$p'_{NL} > 0 \quad H$$

$$p'_{NL} < 0 \quad L$$

$$p'_{NL} > 0 \quad H$$

$\Uparrow$ *upward vertical PGF*

Therefore, the non-linear pressure perturbation due to shear and stretching creates **additional** upward lifting (a pressure gradient force) in the lower atmosphere that enhances the updraft beyond that based on buoyancy alone!

Rule of thumb: 1 mb of VPGF over 1 km ~ same forcing as 3°C of buoyancy.

Relevant Figures: 3.22 and 3.23 in Bluestein, Volume 2

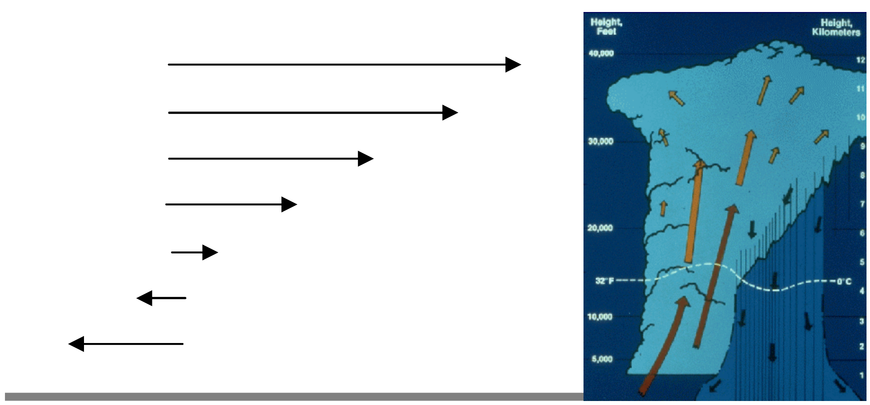
2. Right-moving storms

Consider the linear *p'* term:

$$\nabla^2 p' = -2\rho_o\left(\frac{\partial \bar{u}}{\partial z}\frac{\partial w'}{\partial x} + \frac{\partial \bar{v}}{\partial z}\frac{\partial w'}{\partial y}\right) = -2\rho_o \frac{\partial \vec{V}}{\partial z}\cdot\nabla w', \text{ so}$$

$$p' \propto \frac{\partial \vec{V}}{\partial z}\cdot\nabla w'$$

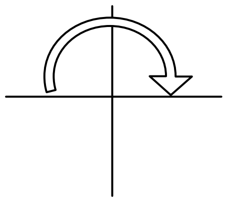
For the sake of simplicity, assume that shear is *unidirectional* with height (for now):



east →

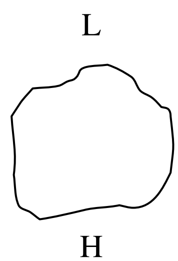
On the western or *upshear* flank of the updraft, $p' \propto \frac{\partial \bar{u}}{\partial z}\frac{\partial w'}{\partial x} > 0$. But on the eastern (*downshear*) flank, $p' < 0$. As we learned before, the pressure perturbation is largest at the mid-levels. Thus, we can expect **new cell growth on the downshear flank** in unidirectional shear.

Now a more complicated scenario. Let the hodograph be clockwise curved, as follows:

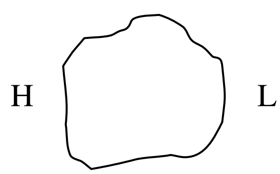


We now must consider *both* terms.

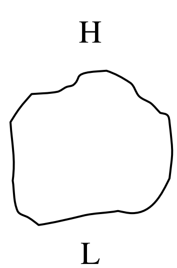
At the low levels, $\frac{\partial \bar{v}}{\partial z}\frac{\partial w'}{\partial y}$ produces



At the mid-levels, $\frac{\partial \bar{u}}{\partial z}\frac{\partial w'}{\partial x}$ produces



At the upper levels, $\frac{\partial \bar{v}}{\partial z}\frac{\partial w'}{\partial y}$ produces



Thus, there is an upward (downward) vertical PGF on the right (left) flank of the storm $\therefore$ new cell growth is enhanced to the right $\therefore$ **rotating updraft becomes a 'right mover.'**

Relevant Figures: 3.21, 3.24, and 3.25 in Bluestein, Vol. 2

Summary

The non-linear shear effect promotes new or continued cell growth on the flanks of the old cells.

The linear effect of tilting biases the cell movement toward the right (left) if the environmental hodograph is curved in a clockwise (counterclockwise) manner.

Unidirectional shear promotes storms that split, with each member of the split pair having components of motion normal to the shear vector and opposite to each other.

New buoyant updrafts form off the axis of the shear because upward-directed perturbation pressure gradients induce upward accelerations there, thus lifting the air to its LFC.

Owing to the low-level convergence associated with upward-moving air, vorticity increases through the stretching of existing vorticity and is advected upward by the updraft.

Right moving storms tend to develop cyclonic rotation, while left-moving storms tend to develop anti-cyclonic rotation. The cyclonic vorticity produced in storms that grow in an environment of clockwise turning shear is **NOT** due to the Earth's rotation.

So, is the pressure perturbation term important? For supercells, yes!!!